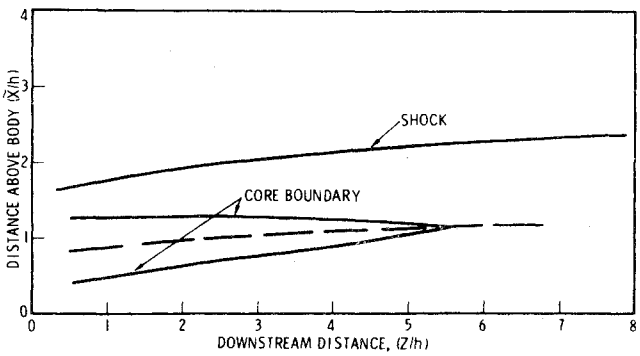

Fig. 2 Illustration of the decay of the core.

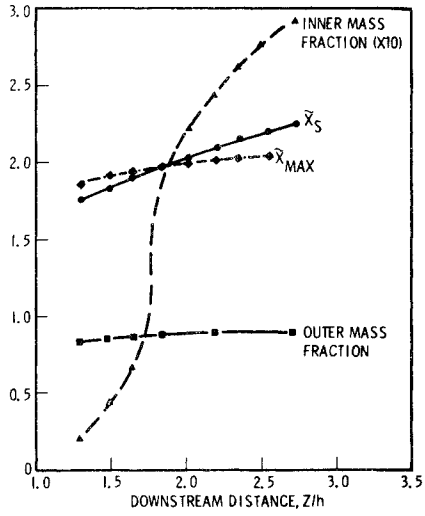

Fig. 3 Mass fractions of air and comparison of shock height and $\bar{x}_{max}$ downstream of injection.

Figure 2 shows results of a typical high Mach number calculation, with the shock location and the upper and lower core boundaries centered about the half-shock height line. At about 5.5 scale lengths downstream of the injection point, the core vanishes and transition to the Gaussian profile occurs.

Other results for the conditions associated with a 6 deg cone at -5 deg angle of attack, 90,000 ft, and $M_\infty = 15$ are shown in Fig. 3. Comparisons of $\bar{x}_s$ and $\bar{x}_{max}$, the $\bar{x}$ value corresponding to γ_{min} and $\bar{y} = 0$ calculated from the Gaussian profiles, are given as a function of z. It can be seen that for the first three steps $\bar{x}_{max}$ is greater than $\bar{x}_s$, implying the existence of a uniform core. Further downstream $\bar{x}_{max} < \bar{x}_s$, implying no potential core remains. Also plotted on the figure are the mass fraction of air in the innermost and outermost of the three radial zones used in the calculation. It can be seen in the upstream region where the core exists that the flow is composed almost entirely of injectant. It is only due to the coarseness of the zones that any air is in the innermost zone. In the vicinity of the core disappearance, the mass fraction of air rises dramatically, signifying the introduction of air into the innermost zone at the end of the injectant core. Rapid changes do not occur in the outer zone when the injectant core vanishes, since there is already considerable mixing.

Remarks and Conclusions

Several aspects of the method just outlined should be noted. The results represent the upper limit on the amount of mixing in any given situation. Less mixing can occur without injectant appearing beyond the shock, but not more.

It may well be found, when experimental data are made available at sufficiently high Mach numbers, that the upper limit of mixing, as found by the method presented here, is the amount that actually occurs. If so, one might conjecture that a change in the physics of the mixing process would affect the shock location; for example, if a large increase in the rate of mixing were to occur, the shock would be required to shift outward to accommodate the wider concentration profiles. A possible mechanism for such a change in the mixing rate is the transition from laminar to turbulent mixing. This would then imply that turbulent-jet shocks would differ—would lie farther from the plate—compared to shocks about laminar transverse jets. This is a point for experimentalists to keep in mind.

The requirement that the profile cutoff be equal to the shock height in the x-z plane, rather than leaving room for an injectant-free shock layer there, may be viewed as somewhat arbitrary. It is true that a shock layer is likely to exist at every point beneath the shock, but secondary flows in both the jet and the shock layer will decrease the shock layer thickness in the x-z plane while increasing it elsewhere. Thus, the error involved is not large, and may, in any case, be limited to the extreme case of a nearly square profile.

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Navier-Stokes Solutions Using Stetter's Method

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Nomenclature

A = area
c = speed of sound
E = total internal energy
F = arbitrary function vector, see Eq. (4)
g = arbitrary function
k = thermal conductivity
N = total number of time steps to convergence
p = pressure
R = time step factor, see Eq. (6)
t = time
T = temperature
u = velocity
U = function vector, see Eq. (4)

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X = axial length
α = adjustable parameter
Δ = incremental value
ρ = density
μ = viscosity

Superscripts

n = time step
( ) = iteration number

Subscript

i = ith spatial value

Introduction

WITH the development of large-scale computers, it has become possible to solve the compressible Navier-Stokes equations for a wide variety of problems; see Ref. 1, which gives an excellent review of the state-of-the-art solutions. Because of the complexity of these solutions, a great deal of effort has been expended in developing more efficient numerical solution techniques, with current emphasis being placed on explicit methods because of their success on vector processing computers. Recently, Ref. 2 studied several of the more popular explicit techniques to determine computational efficiency; however, this reference and other recent studies have not included the method of Stetter,[3] which has been shown to have a greatly expanded region of stability for a simple model problem. Stetter's technique appears to be suited for the rapid (minimal iterative steps) solution of the compressible Navier-Stokes equations. However, since there are no solutions reported in the literature using Stetter's method on systems of equations, the purpose of this paper is to numerically investigate Stetter's technique for a model problem using the compressible Navier-Stokes equations, with the emphasis on the minimum iterative step feature for steady-state solutions and not on time accurate transient solutions.

Governing Equations and Model Problem

Because of the extensive results of Ref. 2, the same model problem will be used in this paper in order to provide a common basis for comparison. This model problem is a quasi-one-dimensional converging-diverging nozzle with the steady-state flowfield initially subsonic, going sonic at the minimum area, passing through a standing normal shock in the diverging section, and exiting the nozzle with subsonic flow. Because of the wide range of flow conditions and the boundary conditions, this model problem presents a rigorous test for numerical methods. The governing equations in divergence form for the time-dependent, quasi-one-dimensional flow of a compressible viscous perfect fluid are

Continuity

$$\frac{\partial(\rho A)}{\partial t}+\frac{\partial(\rho u A)}{\partial X}=0 \tag{1}$$

Momentum

$$\frac{\partial(\rho u A)}{\partial t}+\frac{\partial[A(p+\rho u^2)]}{\partial X}-\frac{\partial}{\partial X}\left(\frac{4\mu}{3}\frac{\partial u}{\partial X}\right)-p\frac{\partial A}{\partial X}=0 \tag{2}$$

Energy

$$\frac{\partial(EA)}{\partial t}+\frac{\partial[Au(E+p)]}{\partial X}-\frac{\partial}{\partial X}\left(\frac{4\mu u}{3}\frac{\partial u}{\partial X}\right)-\frac{\partial}{\partial X}\left(k\frac{\partial T}{\partial X}\right)=0 \tag{3}$$

The coefficient of viscosity μ is obtained from Sutherland's law and the thermal conductivity k is obtained from an assumed constant Prandtl number. In order to facilitate the solution of these equations by Stetter's method, they are rewritten in the following form:

$$\frac{\partial U}{\partial t}=F \tag{4}$$

where

$$U=A\begin{Bmatrix}\rho\\ \rho u\\ E\end{Bmatrix}$$

$$F=\begin{Bmatrix}-\dfrac{\partial(\rho u A)}{\partial X}\\ -\dfrac{\partial[A(p+\rho u^2)]}{\partial X}+\dfrac{\partial}{\partial X}\left(\dfrac{4\mu}{3}\dfrac{\partial u}{\partial X}\right)+p\dfrac{\partial A}{\partial X}\\ -\dfrac{\partial[Au(E+p)]}{\partial X}+\dfrac{\partial}{\partial X}\left(\dfrac{4\mu u}{3}\dfrac{\partial u}{\partial X}\right)+\dfrac{\partial}{\partial X}\left(k\dfrac{\partial T}{\partial X}\right)\end{Bmatrix}$$

As in Ref. 2, the finite-difference representations of the spatial derivatives are given by equally spaced central-difference relations of second-order accuracy.

Solution Technique

In Ref. 3, Stetter gives several predictor-corrector techniques of improved stability. However, in the present investigation, the one technique which was clearly superior to the others is the three-step method given by

$$^{(0)}U_i^{n+1}=U_i^n+\Delta t_i F_i^n \tag{5a}$$

$$^{(1)}U_i^{n+1}=U_i^n+\frac{\Delta t_i}{2}\left(^{(0)}F_i^{n+1}+F_i^n\right) \tag{5b}$$

$$^{(2)}U_i^{n+1}=U_i^n+\frac{\Delta t_i}{2}\left(^{(1)}F_i^{n+1}+F_i^n\right) \tag{5c}$$

$$U_i^{n+1}=\alpha\,^{(1)}U_i^{n+1}+(1-\alpha)\,^{(2)}U_i^{n+1} \tag{5d}$$

In the current application, the time step at each point is given by a constant factor times the local *CFL* condition:

$$R=\Delta x/(|u_i|+c_i) \tag{6}$$

Boundary Conditions

The boundary conditions, from Ref. 2, are as follows. Inflow—the subsonic inflow total pressure and total temperature were specified and held fixed. The third inflow boundary condition was obtained by requiring a zero gradient on static pressure. Outflow—the exit pressure is specified and held fixed and the two other conditions were obtained by requiring a zero gradient on velocity and density.

Discussion of Results

In the original formulation of Stetter's method for a single equation with a real eigenvalue, the value of α which most improved the stability of the three-step method was 0.7. However, the current system of equations using three-point central-difference approximations to the space derivatives of the convection terms leads to imaginary eigenvalues for the system at high Reynolds numbers, and under these conditions the maximum value of α that could be used was $\alpha=0.2$. Both of the preceding conditions are consistent with linearized stability theory. Table 1 gives the number of time steps to convergence (using the same convergence criterion as Ref. 2) for Stetter's method and the four methods studied in Ref. 2. It is immediately obvious that, for the two higher Reynolds

Table 1 Summary of results

	$Re = 45374$		$Re = 11345$		$Re = 2269$	
	$\Delta t/\Delta t_{CFL}$	N	$\Delta t/\Delta t_{CFL}$	N	$\Delta t/\Delta t_{CFL}$	N
MacCormack[a]	0.9	338	0.9	546	0.5	896
Modified hopscotch[a]	1.0	565	0.8	514	0.3	1193
Brailovskaya[a]	1.1	323	1.0	485	0.4	1110
Modified DuFort-Frankel[a]	0.5	990	0.4	1350	...	...
Stetter	2.1	181	2.1	197	0.4	995

[a] Taken from Ref. 2.

number cases, Stetter's method reaches steady state in many fewer time steps than the other four methods. For the lowest Reynolds number case, which is viscous dominated, all of the methods take approximately the same number of steps. It should be noted that Stetter's method requires more work per time step (three evaluations of the space derivatives as opposed to one or two for the other methods); although the CPU time in the present study was not accurately determined, it does appear that the modified hopscotch has a slightly better CPU time. However, there are applications such as Ref. 4, where auxiliary transport calculations require more CPU time than the fluid mechanics, and in such a situation, even though the auxiliary calculations are not made at every time step, the total CPU time for a given problem is reduced by having the fluid mechanics converge faster. Thus, Stetter's method with its ability to use larger marching steps, should find application in such problems.

Concluding Remarks

Stetter's three-step predictor-corrector technique has been used to solve the compressible Navier-Stokes equations for quasi-one-dimensional flow in a converging-diverging nozzle. For the nonviscous dominated case, Stetter's method was clearly superior to the four other currently popular methods in attaining the steady-state solution in the fewest steps. Time steps of slightly greater than two times the CFL limit were obtainable with Stetter's method.

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Integral Equation Formulation for Transonic Flow Past Lifting Wings

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Introduction

THE integral equation method for calculating steady inviscid transonic flows based on small perturbation theory is well known. In general it involves a small amount of numerical work. For the three-dimensional case an integral equation formulation originates from Klunker[1] by the application of Green's theorem. It also gives the expression for the far-field potential for three-dimensional wings at transonic speeds. By means of the small disturbance shock jump conditions, Klunker showed that the surface integral term over the surface of the shock discontinuity does not make any contribution to the potential. Note that Ferrari and Tricomi[2] made a normal shock assumption in the integral equation formulation for two-dimensional nonlifting flows. There, the integral equation was formulated in terms of the velocity component instead of the velocity potential, as was done by Klunker. It is apparent from the work of Klunker that the normal shock assumption, which is one of the drawbacks of the integral equation method, is not necessary if we formulate the integral equation in terms of the velocity potential and subsequently differentiate it to get the same integral equation for the velocity components.[1]

On the other hand, the analytical solution to the transonic small perturbation equation in reduced coordinates was expressed as a nonlinear integral equation by Nörstrud,[3] with the reduced perturbation velocity potential as the unknown function. The purpose of the present Note is to study critically Nörstrud's formulation for three-dimensional transonic lifting wings and to indicate an alternative way of deriving them.

An Alternative Deduction of the Integral Equation Formulation

Using the same notations defined by Niyogi,[4] the integral equation formulated by Nörstrud[3] for reduced U velocity components may be written as follows

$$U(X,\pm 0,Z) = U_P(X,\pm 0,Z) \pm \tfrac{1}{2}[\Delta\bar{U}(X,Z) - \Delta U_P(X,Z)] + \lim_{Y\to 0^{\pm}} \frac{\partial I(X,Y,Z;U)}{\partial X} \tag{1}$$

where $U_P(X,Y,Z) = (\partial/\partial X)\Phi_P(X,Y,Z)$; Φ_P the solution of Laplace equation

$$\Phi_{XX} + \Phi_{YY} + \Phi_{ZZ} = 0 \tag{2}$$

satisfies the same boundary condition as $U(X,Y,Z)$. $I(X,Y,Z;U)$ represents the volume integral

$$I(X,Y,Z;U) = \frac{-1}{4\pi}\int\int\int_{-\infty}^{\infty} \Phi_\xi(\xi,\eta,\zeta)\Phi_{\xi\xi}(\xi,\eta,\zeta)\frac{1}{R}\,d\xi d\eta d\zeta \tag{3}$$

It has been shown in Ref. 4 that the two-dimensional equations corresponding to Eq. (1) are not independent. Proceeding in a similar way, the same conclusion can be drawn for the present three-dimensional case also. Alternatively, following Nörstrud,[5] if the symmetric and antisymmetric parts of $U(X,Y,Z)$ and $U_P(X,Y,Z)$ are defined by U^+, U^-, and U_p^+, U_p^-, respectively, then Eqs. (1) becomes

$$U^+ + U^- = U_p^+ + U_p^- + U^- - U_p^- + \lim_{Y\to 0^+}(\partial I/\partial X)$$

or

$$U^+ = U_p^+ + \lim_{Y\to 0^+}(\partial I/\partial X) \tag{4}$$

and

$$U^+ - U^- = U_p^+ - U_p^- - U^- + U_p^- + \lim_{Y\to 0^-}(\partial I/\partial X)$$

or

$$U^+ = U_p^+ + \lim_{y\to 0^-}(\partial I/\partial X) \tag{5}$$

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